MINKOWSKI LINEAR SET OPERATORS

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Аннотация. В статье излагается концепция возможных комбинаций Минковского множеств точек, которые можно определить как обобщение операций суммы Минковского двух множеств точек и произведения Минковского двух множеств точек в Эвклидовом пространстве $\boldsymbol{E}^n$. Линейная и матричная комбинация суммы Минковского и произведения Минковского двух множеств точек, а также арифметическая комбинация Минковского трех множеств точек представлены и проиллюстрированы на примерах.

Abstract. Paper brings few ideas about a concept of possible Minkowski combinations of point sets, which can be defined as generalisation of set operations Minkowski sum and Minkowski product of two point sets in the Euclidean space $\boldsymbol{E}^n$. Minkowski sum linear and matrix combination, Minkowski product linear and matrix combination, and Minkowski arithmetic combination of point sets are introduced and illustrated on examples.

Key Words: Minkowski sum and product, Minkowski linear, matrix and arithmetic combination of point sets

1. Minkowski linear combinations of two point sets

Concept of Minkowski linear combination of two point sets is an analogy of the well defined concept of a linear combination of two vectors. It can be defined on the basis of the scalar multiple of a vector. Let us introduce the operation of a scalar multiple of a point set.

Definition 1.1. Multiple of a point set $A \subset \boldsymbol{E}^n$ by scalar $k \in R$ is point set $A_k \subset \boldsymbol{E}^n$ with elements in all such points of the space, whose positioning vectors are k-multiples of positioning vectors of all points of the set A

$$A_k = k.A = \{k.m : m \in A, m \mapsto \mathbf{m} \Rightarrow k.m \mapsto k.\mathbf{m}\}, k \in R. \tag{1.1}$$

Set A_k is denoted also as dilatation of set A by a non-zero scalar $k \in R$, and it can be interpreted geometrically as a homothetic scaled image of the point set A in the homothety with centre in the reference point O and with scaling coefficient k. Symmetric configuration of points in the set A is therefore an invariant property that appears also in the configuration of points in the k-multiple image - point set A_k.

Minkowski linear combination of two point sets can be defined as generalisation of the operation of Minkowski sum of point sets.

Definition 1.2. Minkowski sum linear combination of two point sets A and B in the space $\boldsymbol{E}^n$ is point set C in the space $\boldsymbol{E}^n$ defined as follows

$$C = k.A \oplus l.B = A_k \oplus B_l = \{k.a + l.b, a \in A, b \in B\}, k, l \in R. \tag{1.2}$$

Consequence 1.1. Minkowski sum linear combination of two congruent point sets $A = B$ in the space $\boldsymbol{E}^n$ is set

$$C \supseteq A_{k+l}, k, l \in R. \tag{1.3}$$

Proof: $C = k.A \oplus l.A = A_k \oplus A_l = \{k.a + l.b, a \in A, b \in A\} =$

$$= \{k.a + l.a, a \in A\} \cup \{k.a + l.b, a \in A, b \in A, a \neq b\} =$$
$$= \{(k + l).a, a \in A\} \cup \{k.a + l.b, a \in A, b \in A, a \neq b\} =$$
$$= (k + l).A \cup \{k.a + l.b, a \in A, b \in A, a \neq b\} \supseteq A_{k+l}$$

Consequence 1.2. Minkowski sum of a point set A in the space $\boldsymbol{E}^n$ with itself is the point set

$$A \oplus A \supseteq A_2 . \tag{1.4}$$

Proof: $A \oplus A = \{a + b, a \in A \wedge b \in A\} = \{a + a, a \in A\} \cup \{a + b, a \in A, b \in A, a \neq b\} =$
$= \{2a, a \in A\} \cup \{a + b, a \in A, b \in A, a \neq b\} \supseteq 2.A = A_2$.

Several illustrations of Minkowski linear combinations of two discrete point sets are presented in Fig. 1, while Minkowski linear combinations of two equally parameterized planar curve segments are in Fig. 2. In Fig 3, Minkowski linear combinations of two curve segments in 3-dimensional space are visualised as surface patches.

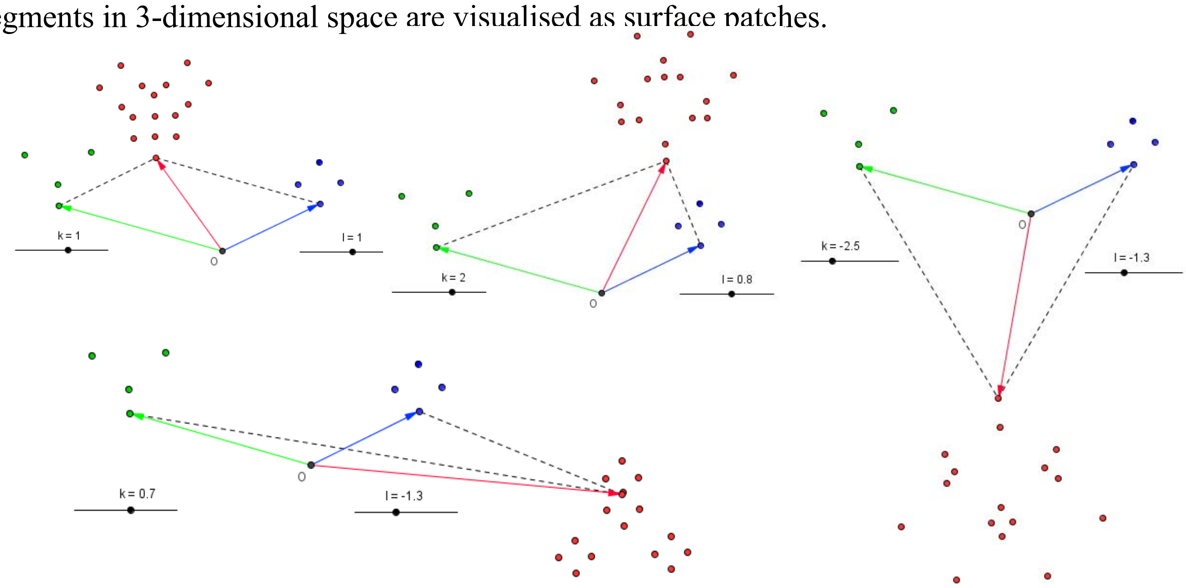

Fig. 1 Minkowski sum linear combinations of two discrete point sets

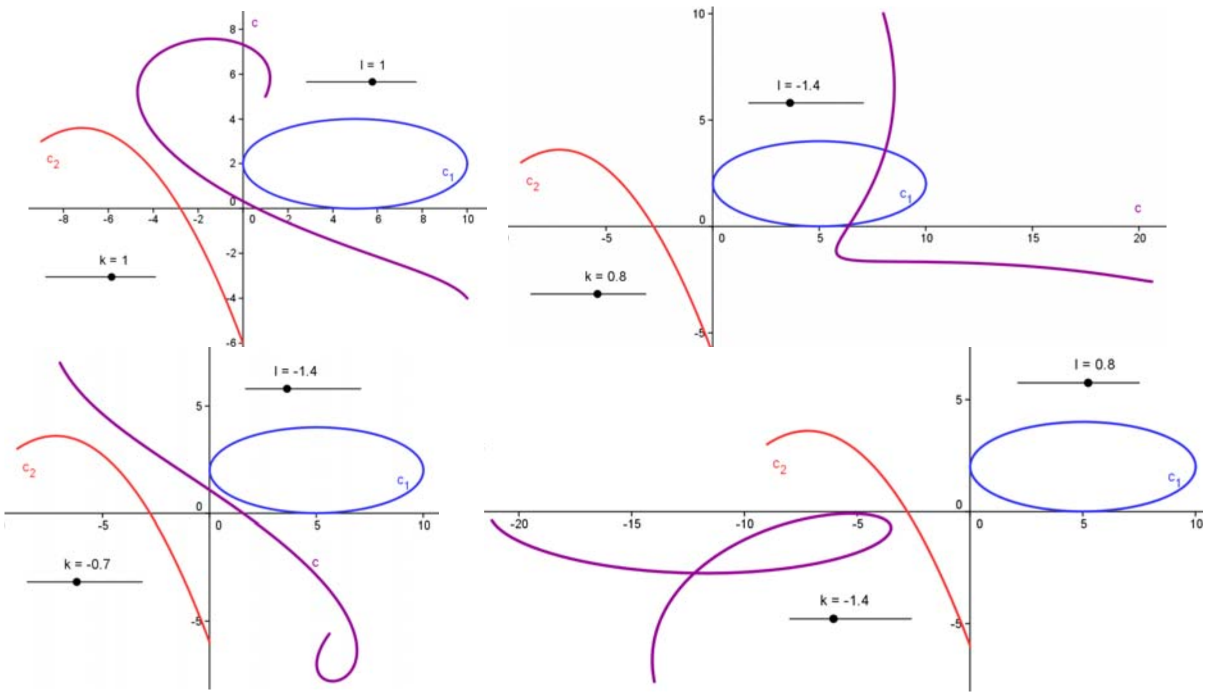

Fig. 2 Minkowski sum linear combinations of two planar curve segments

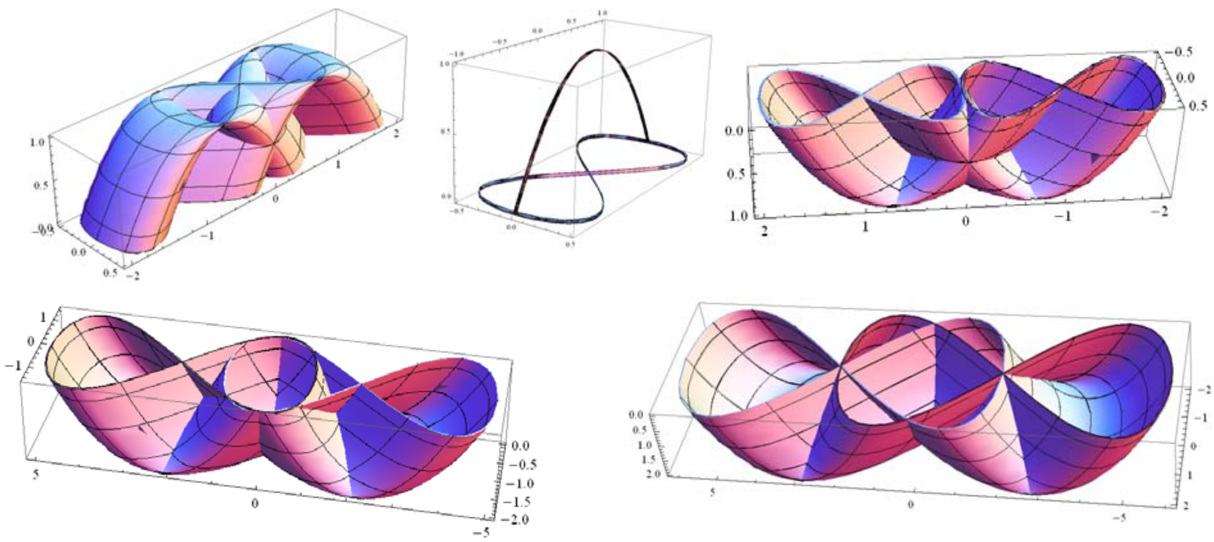

Fig. 3 Minkowski sum linear combinations of two curve segments in 3D

Definition 1.3. Minkowski product linear combination of two point sets A and B in the space $\boldsymbol{E}^n$ is point set C in the space $\boldsymbol{E}^n$ defined as

$$C = k.A \otimes l.B = A_k \otimes B_l = \{k.a \wedge l.b, a \in A, b \in B\}, k, l \in R. \tag{1.5}$$

Consequence 1.3. Minkowski product linear combination $k.A \otimes l.B$ of two point sets A and B in the space $\boldsymbol{E}^n$ is the $k.l$ multiple of Minkowski product $A \otimes B$ of point sets A and B

$$k.A \otimes l.B = A_k \otimes B_l = (A \otimes B)_{k.l}, k, l \in R. \tag{1.6}$$

Proof: $k.A \otimes l.B = A_k \otimes B_l = \{k.a \wedge l.b, a \in A, b \in B\} =$

$$= \{k.l(a \wedge b), a \in A, b \otimes B\} = k.l.(A \otimes B) = (A \otimes B)_{k.l}.$$

Illustrations of Minkowski product and Minkowski product linear combinations of the same two curve segments as in Fig. 3 for Minkowski sum linear combinations are presented in Fig. 4.

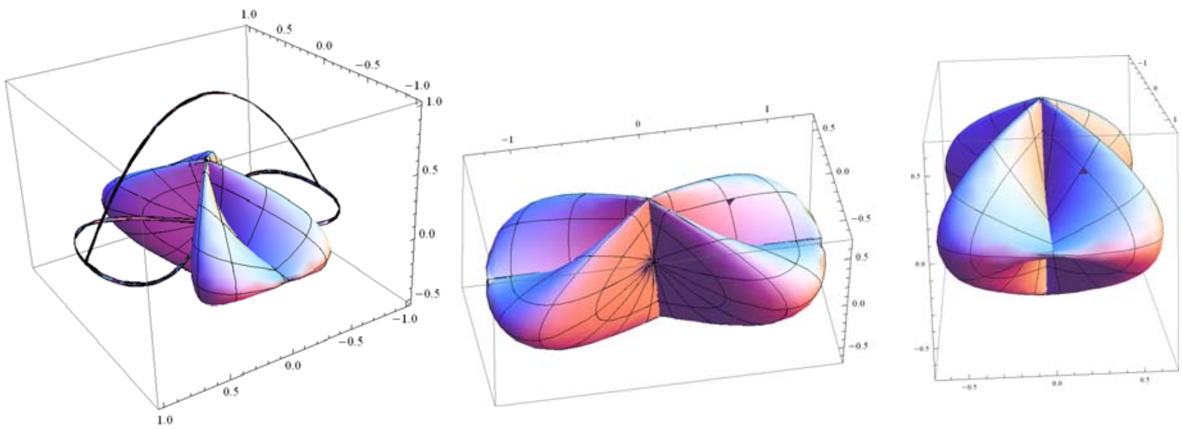

Fig. 4 Views of Minkowski product linear combination of two curve segments in 3D

2. Minkowski matrix combinations of two point sets

Operation of matrix multiple of a point set in $\boldsymbol{E}^n$ by a suitable matrix can be defined by means of matrix multiplication. Positioning vectors of points in the set in $\boldsymbol{E}^n$ are matrices of type $1 \times n$, operation of matrix multiplication can be therefore defined for square matrices of type $n \times n$. Concept of Minkowski sum and product matrix combination can be introduced.

Definition 2.1. Let $\mathbf{M} = (a_{ij})_{i,j=1,\dots,n}$ be a regular square matrix of rank n and A be a non-zero point set in the space $\boldsymbol{E}^n$. Multiple of matrix $\mathbf{M}$ and set A is point set $A_{\mathbf{M}}$ in the space $\boldsymbol{E}^n$, defined as

$$A_{\mathbf{M}} = A.\mathbf{M} = \{m.\mathbf{M} : m \in A, m \mapsto \mathbf{m} \Rightarrow m.\mathbf{M} \mapsto \mathbf{m}.\mathbf{M}\}. \tag{2.1}$$

Multiple of a point set A and matrix $\mathbf{M}$ can be geometrically interpreted as follows: set A_M is the image of point set A in the linear geometric transformation of the space $\mathbf{E}^n$ determined by the respective regular square matrix $\mathbf{M}$. Configuration of points in the set A is mapped in the given transformation into a new configuration, in which e.g. symmetry could be no more invariant.

Definition 2.2. Let A, B be point sets in the space $\boldsymbol{E}^n$ and $\mathbf{M}$, $\mathbf{N}$ be regular square matrices of rank n with real entries. Minkowski sum matrix combination of point sets A, B is point set C in space $\boldsymbol{E}^n$

$$C = A.\mathbf{M} \oplus B.\mathbf{N} = A_{\mathbf{M}} \oplus B_{\mathbf{N}} = \{a.\mathbf{M} + b.\mathbf{N}, a \in A, b \in B\}. \tag{2.2}$$

Consequence 2.1. Minkowski sum linear combination of point sets is a special case of Minkowski sum matrix combination, in which both matrices $\mathbf{M}$ and $\mathbf{N}$ are diagonal matrices with non-zero entries, except those on major diagonals that are equal numbers in respective matrices, k in matrix $\mathbf{M}$ and l in matrix $\mathbf{N}$.

Proof: Matrix multiples $A.\mathbf{M}$ and $B.\mathbf{N}$ of sets A, B and diagonal matrices $\mathbf{M}$, $\mathbf{N}$ are

$$A.\mathbf{M} = \{a.\mathbf{M}, a \in A\},\ B.\mathbf{N} = \{b.\mathbf{N}, b \in B\},$$

while for any element $a \in A$, or $b \in B$ the following is valid

$$a.\mathbf{M} = a.\begin{pmatrix} k & 0 & \dots & 0 \\ 0 & k & \dots & 0 \\ & \dots & & \\ 0 & 0 & \dots & k \end{pmatrix} = k.a,\ b.\mathbf{N} = a.\begin{pmatrix} l & 0 & \dots & 0 \\ 0 & l & \dots & 0 \\ & \dots & & \\ 0 & 0 & \dots & l \end{pmatrix} = l.b$$

$$A.\mathbf{M} = \{k.a, a \in A\} = A_k,\ B.\mathbf{N} = \{l.b, b \in B\} = B_l$$

Then, Minkowski sum matrix combination of point sets A and B can be written as

$$\begin{aligned} A_{\mathbf{M}} \oplus B_{\mathbf{N}} &= A.\mathbf{M} \oplus B.\mathbf{N} = \{a.\mathbf{M} + b.\mathbf{N}, a \in A, b \in B\} = \\ &= \{k.a + l.b, a \in A, b \in B\} = A_k \oplus B_l = k.A \oplus l.B \end{aligned}$$

Examples of Minkowski sum matrix combinations of two finite point sets are in Fig. 5, on Fig. 6 Minkowski sum matrix combinations of equally parameterized planar curves are presented, while in Fig. 7 Minkowski sum matrix combinations of two space curve segments from illustrations in Fig 3 are given.

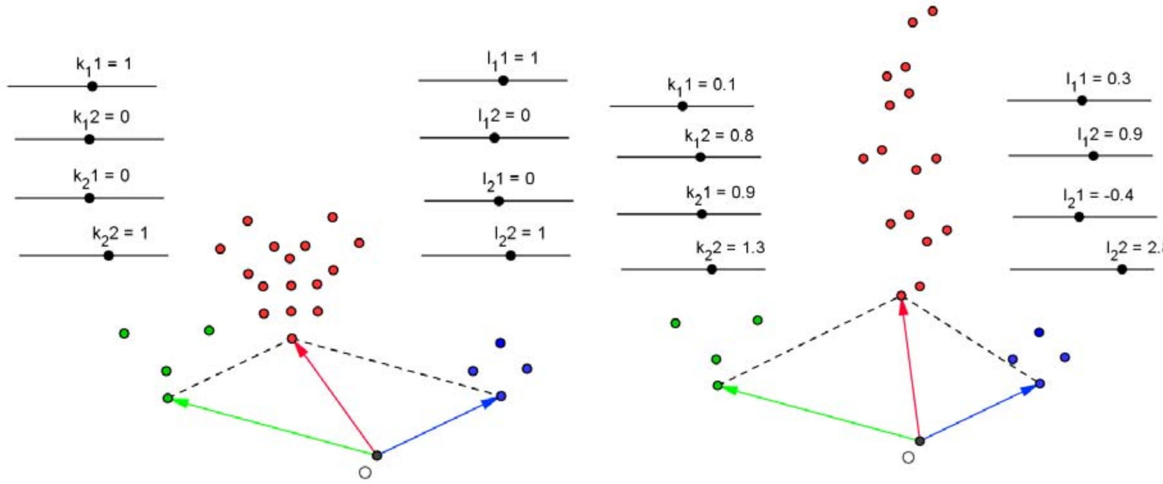

Fig. 5 Minkowski sum matrix combinations of two discrete point sets

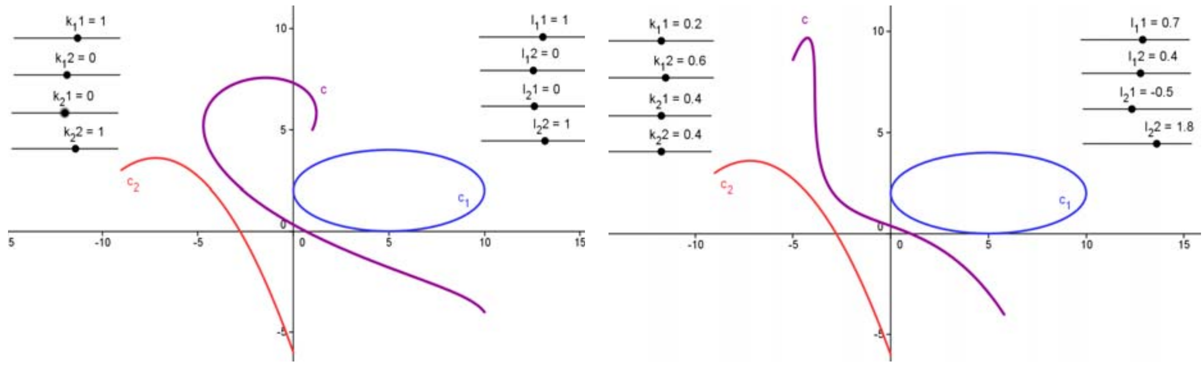

Fig. 6 Minkowski sum matrix combinations of two planar curve segments

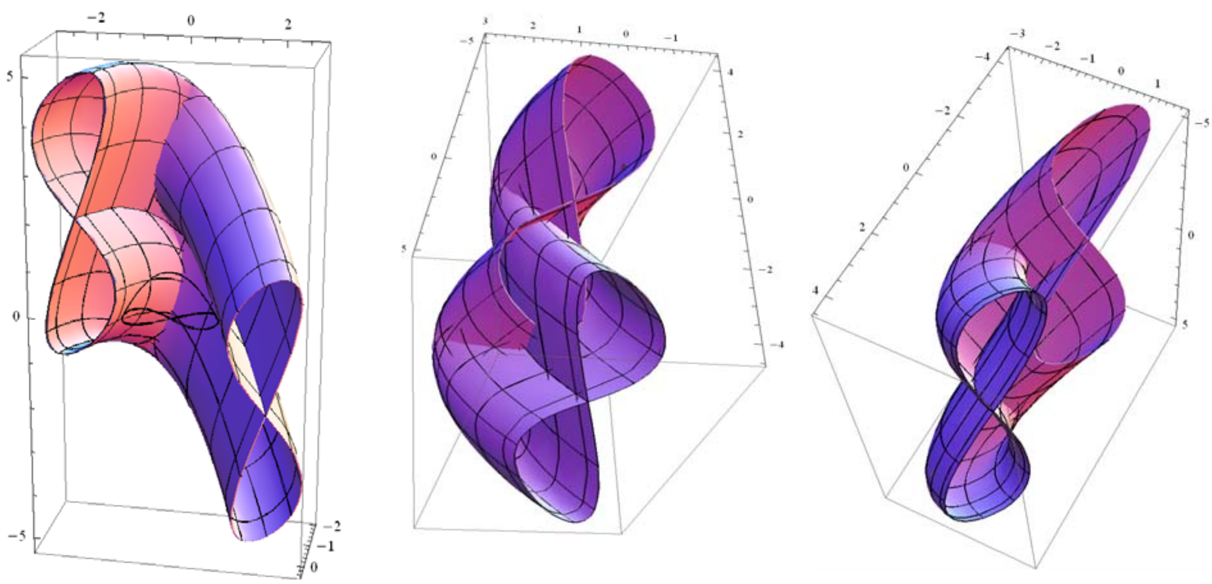

Fig. 7 Minkowski sum matrix combinations of two curve segments in 3D

Definition 2.3. Let A, B be point sets in the space $\boldsymbol{E}^n$ and $\mathbf{M}$, $\mathbf{N}$ be regular square matrices of rank n with real entries. Minkowski product matrix combination of point sets A, B is point set C in space $\boldsymbol{E}^n$

$$C = A.\mathbf{M} \otimes B.\mathbf{N} = A_{\mathbf{M}} \otimes B_{\mathbf{N}} = \left\{ a.\mathbf{M} \wedge b.\mathbf{N}, a \in A, b \in B \right\}. \qquad (2.3)$$

Consequence 2.2. Minkowski product matrix combination $A.\mathbf{M} \otimes B.\mathbf{N}$ of two point sets A and B in the space $\boldsymbol{E}^n$ is the $\mathbf{M.N}$ matrix multiple of Minkowski product $A \otimes B$ of point sets A and B

$$A.\mathbf{M} \otimes B.\mathbf{N} = A_{\mathbf{M}} \otimes B_{\mathbf{N}} = (A \otimes B)_{\mathbf{M.N}}, k, l \in R. \qquad (1.5)$$

Proof: $A.\mathbf{M} \otimes B.\mathbf{N} = A_{\mathbf{M}} \otimes B_{\mathbf{N}} = \left\{ a.\mathbf{M} \wedge l.\mathbf{N}, a \in A, b \in B \right\} =$

$$= \left\{ \mathbf{M.N}(a \wedge b), a \in A, b \otimes B \right\} = \mathbf{M.N}(A \otimes B) = (A \otimes B)_{\mathbf{M.N}}.$$

In Fig. 8, views of three surfaces are presented, which are generated as Minkowski product matrix combinations of the same two curve segments that are chosen in examples of Minkowski sum and product linear combinations viewed in Fig. 3 and Fig. 4, and Minkowski sum matrix combinations in Fig. 7.

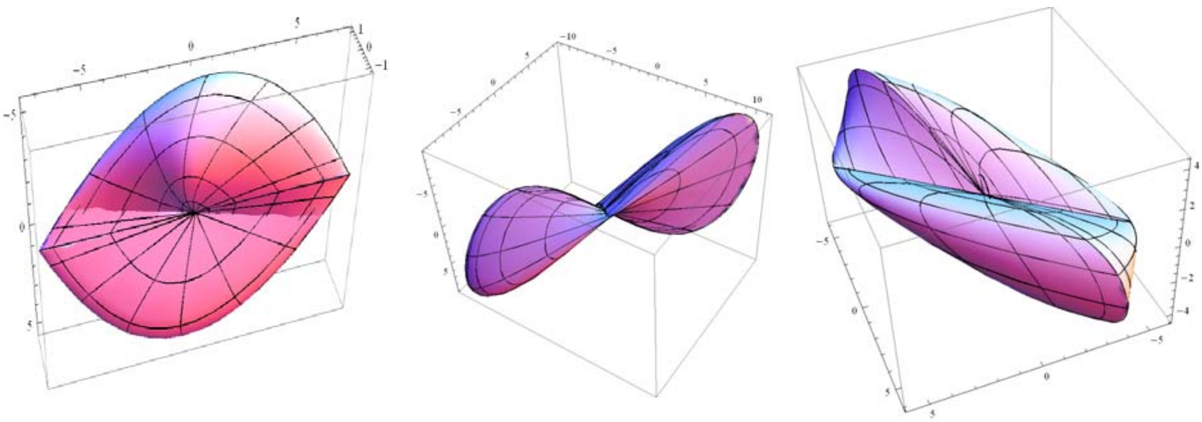

Fig. 8 Minkowski product matrix combinations of two curve segments in 3D

3. Minkowski arithmetic combination of three point sets

Concept of Minkowski arithmetic combination of three sets can be introduced, based on Minkowski sum and .

Definition 3.1. Let A, B, C be point sets in the space $\boldsymbol{E}^n$ and k, l, h be real numbers. Minkowski arithmetic combination of point sets A, B, C is point set W in space $\boldsymbol{E}^n$ defined by trhe following formula

$$\begin{aligned} W &= (k.A \oplus l.B) \otimes h.C = (A_k \oplus B_l) \otimes C_h = \\ &= \left\{ (k.a + l.b) \wedge h.C, a \in A, b \in B, c \in C \right\} \end{aligned} \qquad (3.1)$$

Consequence 3.1. Minkowski arithmetic combination $(k.A \oplus l.B) \otimes h.C$ of three point sets A, B and C in the space $\boldsymbol{E}^n$ can be represented as the h-multiple of Minkowski sum of Minkowski product linear combinations of sets A and C and B and C

$$(k.A \oplus l.B) \otimes h.C = (A_k \oplus B_l) \otimes C_h = ((A \otimes C)_k \oplus (B \otimes C)_l)_h, h, k, l \in R. \qquad (3.2)$$

Proof: $(k.A \oplus l.B) \otimes h.C = \{(k.a + l.b) \wedge h.c, a \in A, b \in B, c \in C\} =$

$$= \{(k.a \wedge h.c) + (l.b \wedge h.c), a \in A, b \in B, c \in C\} =$$
$$= (k.A \otimes h.C) \oplus (l.B \otimes h.C) = k.h(A \otimes C) \oplus l.h(B \otimes C) =$$
$$= (A \otimes C)_{k.h} \oplus (B \otimes C)_{l.h} = ((A \otimes C)_k \oplus (B \otimes C)_l)_h$$

Consequence 3.2. Minkowski arithmetic combination of three point sets A, B and C in the space $\boldsymbol{E}^n$ for constants $k = l = h$ multiple of Minkowski product $A \otimes B$ of point sets A and B

$$(k.A \oplus k.B) \otimes k.C = ((A \otimes C) \oplus (B \otimes C))_{k^2} \qquad (3.3)$$

Proof: $(k.A \oplus k.B) \otimes k.C = \{(k.a + k.b) \wedge k.c, a \in A, b \in B, c \in C\} =$

$$= \{k.(a+b) \wedge k.c, a \in A, b \in B, c \in C\} =$$
$$= \{k^2.((a+b) \wedge c), a \in A, b \in B, c \in C\} =$$
$$= \{k^2.((a \wedge c) + (b \wedge c)), a \in A, b \in B, c \in C\} =$$
$$= k^2.((A \otimes C) \oplus (B \otimes C)) = ((A \otimes C) \oplus (B \otimes C))_{k^2}$$

Minkowski arithmetic combination of three equally parameterized curve segments is a curve segment, examples are visualised in Fig 9. The same three curve segments viewed in Fig. 9 on the left are used for illustration of Minkowski arithmetic combinations in which two curves are equally parameterized. The resulting surface patches can be seen in Fig. 10.

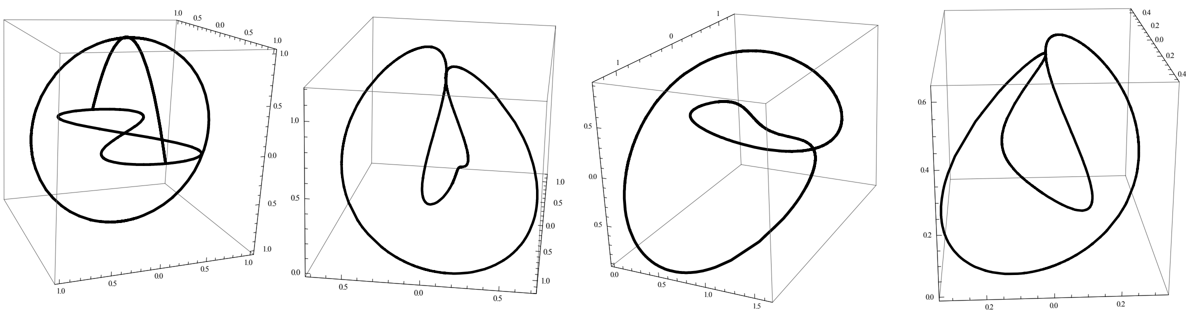

Fig. 9 Minkowski arithmetic combinations of three curve segments

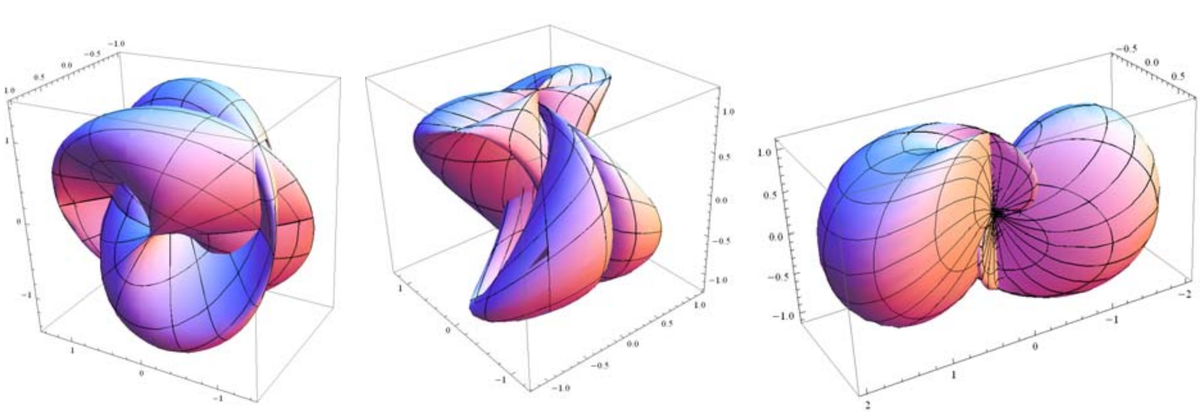

Fig. 10 Minkowski arithmetic combinations of three curve segments in 3D

Acknowledgement

This paper was supported by the grant VEGA No. 1/0230/11.

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